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Damage spreading in the Bak–Sneppen model

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Abstract. We explain the results recently obtained for the damage-spreading behaviour in the Bak–Sneppen (BS) model (Tamarit *et al* 1998 *Eur. Phys. J.* B **1** 545). We do this by relating the BS model to a much simpler one, which includes many features of the BS model and provides a clear explanation for the occurrence of power-law growth of the distance.

1. Introduction

A great deal of evidence has been put forward in recent years for the appearance of criticality in nature. From biological evolution [2] to earthquakes [3], from surface growth [4] to fluid displacement in porous media [5], a wide variety of phenomena exhibit scale invariance in both, space and time. Scale invariance means that the correlation length in these systems is infinite and consequently, a small (local) perturbation can produce a global (maybe even drastic) effect. This possibility leads naturally to the study of the sensitivity to perturbations in critical systems.

To study the propagation of local perturbations (*damage spreading*) in critical systems one can borrow a technique from dynamical systems theory. Let us consider two copies of the same dynamical system (let us say, for instance, a one-dimensional (1D) map), with slightly different initial conditions. By following the dynamics of both copies and studying the evolution in time of the 'distance' $d(t)$ between them, it is possible to quantify the effect of the initial perturbation. Indeed, assuming the distance $d(t)$ grows exponentially, and defining the Lyapunov exponent λ via

$$d(t) = d_0 \exp(\lambda t) \tag{1}$$

three different behaviours can be distinguished, corresponding to λ being either positive, negative or zero. The case $\lambda > 0$ corresponds to the so-called *chaotic* systems, where the extremely high sensibility to initial conditions leads to exponentially diverging trajectories. The case $\lambda < 0$, instead, characterizes those systems in which the dynamics has an attractor (such as a fixed point or a limit cycle) and any initial perturbation is 'washed out' with exponential rapidity.

The boundary case, $\lambda = 0$, admits, in turn, a whole class of functions $d(t)$. We will only concern ourselves here with the case (common in critical systems)

$$d(t) \sim t^{\alpha} \tag{2}$$

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where α is some exponent, characteristic of the system. The case $\alpha > 0$ corresponds to weak sensitivity to initial conditions while $\alpha < 0$ corresponds to weak insensitivity to initial conditions [6]. As an example, the reader is referred to [6], where this analysis is performed for the logistic map at its critical point [7]. Moreover, in [6], the behaviour described by equation (2) has been related to the non-extensivity of the entropy proposed in [7].

In a recent paper [1], a similar analysis was performed on the Bak–Sneppen (BS) model [2]. Originally proposed to describe ecological evolution, this model has been paid a great deal of attention due to its simplicity and the fact that it exhibits self-organised criticality [8]. Schematically, the BS model is defined on a lattice where at each timestep one site is chosen, namely the one that fulfils a global constraint (minimum in some phase space)†. This site is defined as the *active site*. This dynamics leads to a non-Markovian process where the activity, i.e. the position on the lattice of the active site, jumps on the lattice following a (correlated) Lévy walk. Its critical properties allow us to describe its behaviour under perturbations via equation (2), with

$$\alpha = 0.32. \tag{3}$$

In this paper, we explain the results obtained in [1], concerning the behaviour under perturbations of the BS model. By comparing the BS model with a much simpler model, we are able to explain the appearance of a power-law growth of the Hamming distance $D(t)$. As we shall see, the particular stationary distribution of the variable does not play any role in the determination of the exponent α in equation (2). What matters is, instead, the kind of Lévy walk involved and the strength of the correlations. The extremal dynamics (i.e. the choice of some extremal value in the system) is by definition non-local and as such has (until now) prevented any analytic treatment of the model. This dynamics leads to avalanches of causally connected events the distribution of which is scale free and therefore described by a power law. Actually, as will be analysed later on, one can simplify the dynamics by considering only some of the features of the model. Our program will therefore be the one of considering simple models with a Lévy walk behaviour of the activity. For these models we can analytically predict the exponent α thus providing a clear explanation for this behaviour, so far only studied numerically for the BS model [1]. Finally, with the help of the theory developed on the basis of these simple models, we will clearly show where the correlations enter and what their effect is in terms of values of α.

2. Damage-spreading in the ring

Let us start by considering a lattice of N sites on a 1D ring R_1. To each site j we assign a random number f_j, extracted from a uniform distribution between 0 and 1. We then consider a 'replica' R_2, in which we introduce a perturbation by exchanging the positions of the values of f_{k_1} and f_{k_2}. We define as active the sites k_1 in R_1 and (the randomly chosen site) k_2 in R_2, namely those sites at which we have the same value of f. It is clear that, from a statistical point of view, both system R_1 and replica R_2 are described by the same distribution function. This prescription corresponds, in a suitably defined phase space, to a small difference in the initial conditions between R_1 and R_2 (see equation (7) below). Moreover, this procedure of finding an active site and exchanging its position with another site taken at random along the lattice, corresponds exactly to the one proposed in [1] for the BS model (other definitions for the initial perturbation are considered elsewhere [10, 11]). The dynamics on the ring(s) is defined as follows. At each timestep, an integer random number x between 1 and N is chosen.

† Unless otherwise stated, all the results presented here correspond to a 1D lattice.

Bearing in mind that our rings have periodic boundary conditions, the positions of the new active sites at time $t + 1$ is then given by

$$k_1(t+1) = k_1(t) + x \tag{4}$$

$$k_2(t+1) = k_2(t) + x \tag{5}$$

on the rings R_1 and R_2, respectively. On these active sites, the value of the variables f is changed, assigning to both of them the same random number (this corresponds to the choice of the same sequence of random numbers in [1] or to the same thermal noise in usual damage spreading calculations [9]).

As both system R_1 and replica R_2 evolve, we compute the Hamming distance, namely

$$D(t) = \frac{1}{N}\sum_{j=1}^{N} |f_j^1 - f_j^2|. \tag{6}$$

Since this quantity has strong fluctuations, we will consider the average $\langle D(t)\rangle$, over realizations. In particular, at $t = 1$, the average (initial) distance $\langle D(1)\rangle$ can be obtained from equation (6),

$$\langle D(1)\rangle = \frac{2}{N}\int_0^1 \mathrm{d}f_1 \mathrm{d}f_2 \eta_1(f^1)\eta_2(f^2)\,|f^1 - f^2| \tag{7}$$

where η_i is the distribution function (at $t = 1$) for the variable $f^i \in \mathbb{R}_i$. In this toy model, both distributions $\eta_i(f)$ are the same, namely a uniform distribution in $f^i \in \mathbb{R}_i$. A simple computation yields

$$\langle D(1)\rangle = \frac{2}{3N}. \tag{8}$$

Applying a similar procedure, one can verify that for $1 \ll t \ll N$ the distance grows linearly. Indeed, let us define $\sigma(t)$ as the averaged number of different sites covered in one copy of the system at time $1 \ll t \ll N$. Then, at time t only $\sigma(t)$ sites have been changed and these are the only ones that contribute to distance. From this consideration it follows that

$$\langle D(t)\rangle = \langle D(1)\rangle\,\sigma(t) \tag{9}$$

where the fact that both replica contribute to the distance on the same footing is taken into account in equation (8). In the case of the ring, if $N \gg 1$ and $1 \ll t \ll N$ the system will always choose a new site at each timestep, and therefore $\sigma(t) \sim t$ (note that, in the 1D case for this dynamics, this is the fastest possible growth of $\langle D(t)\rangle$). This behaviour stops at times $t \sim N$ where a crossover to a saturation regime appears. Clearly, after $t \propto N$ timesteps each site of the lattice has been covered at least once. For $t \gg N$, almost all the lattice sites have been covered and the two strings are made of the same random numbers shifted by $k_2 - k_1$. Thus, the distance reaches a plateau, independent on the size N of the system, given by

$$\langle D(t \to \infty)\rangle = \int_0^1 \mathrm{d}f^1 \mathrm{d}f^2 \rho_1(f^1)\rho_2(f^2)|f^1 - f^2| \tag{10}$$

where ρ_i is the normalized distribution function (at $t = \infty$) for the variable $f^i \in \mathbb{R}_i$. In equation (10), for the particular case of the ring the distributions in the integral are given by $\rho_i = \eta_i$. Applying equation (10) to R_1 and R_2 we finally obtain

$$\lim_{t\to\infty} \langle D(t)\rangle = \tfrac{1}{3}. \tag{11}$$

Note that the same result can be obtained from equations (8), (9) once $\sigma = N$ is inserted. To have an initial distance independent of the lattice size, we consider the ratio $\langle D(t)\rangle/\langle D(1)\rangle$. For this ratio, however, the value of the plateau depends linearly on N, i.e.

$$\frac{\langle D(\infty)\rangle}{\langle D(1)\rangle} = \frac{N}{2}. \tag{12}$$

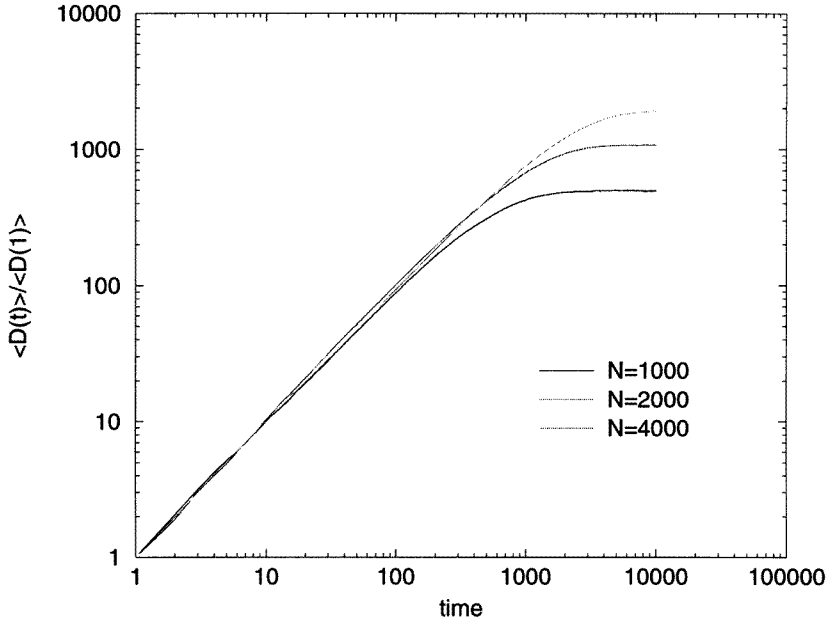

Figure 1. Distance $D(t)$ between the two replicas for three different sizes N for the uncorrelated ring model.

In figure 1 we show the evolution in time of the ratio $\frac{\langle D(t)\rangle}{\langle D(1)\rangle}$, averaged over many realizations, for different lattice sizes. The plateau reached for $t \to \infty$ depends on N according to equation (12). The exponent $\alpha = 1$ obtained for this case from equation (9) can also be obtained numerically with great accuracy. From a physical point of view, this power-law behaviour originates in the ability of the system to cover the lattice. As we have seen, the activity can jump anywhere on the lattice with probability $1/N$. Thus the number of sites j with the same f_j decreases linearly with time and $\langle D(t)\rangle$ increases linearly with time. As a consequence, the time τ needed to reach the plateau scales with the lattice size as $\tau \sim N$.

Bearing in mind our goal of modelling the behaviour of the BS model, let us now consider the case of Lévy walk-type activity jumps along the lattice. More precisely, the length x of any jump is extracted from a power-law distribution function, namely

$$P(x) = (\beta - 1)x^{-\beta} \tag{13}$$

where the minimum jump is $x = 1$ and the jump can be to the left or to the right†.

As before, the position of the new active site is obtained by jumping x sites from the present one, i.e. the position of the new active site will be given by equations (4), (5), where each copy of the system has its own x. Thus, the values of x are uncorrelated between the two copies of the system. The new values of f assigned to the active sites are the same. This choice results in a different behaviour at the saturation regime.

In the random walk limit, $\beta \gg 1$ in equation (13), the distance equation (6), can be easily computed by considering equation (9) together with the fact that $\sigma \sim t^{1/2}$. This calculation yields

$$\langle D(t)\rangle \sim t^{1/2}. \tag{14}$$

† For finite N, due to the periodic boundary conditions, one applies the restriction $x \leqslant \frac{N}{2}$.

In the general case $\beta > 1$, there is still a power-law growth of the distance (6) for intermediate times $1 \ll t \ll N$. The exponent in equation (2) can be obtained using the fact that the mean-square distance σ^2 covered by a Lévy walk behaves like

$$\sigma^2(t) \sim \begin{cases} t^2 & (1 < \beta \leqslant 2) \\ t^{4-\beta} & (2 < \beta < 3) \\ t \log t & (\beta = 3) \\ t & (\beta > 3) \end{cases} \tag{15}$$

in the long-time limit. As a consequence, one can also compute the so-called dynamical exponent z defined through $\tau \sim N^z$ where τ is the time needed for the distance (actually the ratio $\frac{\langle D(t)\rangle}{\langle D(1)\rangle}$) to reach the plateau. This time is given by the time needed to cover all the lattice sites, if finite size effects are not counted in. Comparison between equations (2), (9), (15) yields $z = \frac{1}{\alpha}$.

In this simple model we have excluded any kind of correlation between the values of x extracted from equation (13) and between the two replicas at $t = 1$. Indeed, the dynamics is given by a generalized random walk and therefore the power-law behaviour of the growth is not related to the statistical properties of the system. This is in fact the idea behind our toy model: we use it as a 'black box', not knowing what happens inside, we are only able to observe a Lévy walk behaviour of the activity. Our model is, by conception, a trivial system that has only one purpose i.e. that of showing what the consequences are, in the context of damage spreading, of a power-law behaviour of the activity like the one observed in the BS model.

As we shall see in the next section, the non-trivial properties of the self-organised critical systems are hidden in the value of the exponent α in equation (2). Furthermore, the averaging procedure leading from equation (6) to equation (9) plays a very important role in these non-trivial systems.

Before moving onto the analysis of the BS model, let us discuss in more detail which terms are contributing to the computation of $\langle D(t)\rangle$ via equation (9). In the ring, we have defined $\langle D(t)\rangle$ by considering the behaviour of $\sigma(t)$, which is a physical quantity related only to the behaviour of the activity in one single system. In general, considering that the two replica might be correlated, we need to update equation (9) to

$$\langle D(t)\rangle = \langle D(1)\rangle\, \bar{n}_{cov}(t) \tag{16}$$

where $\bar{n}_{cov}(t)$ is the average number of *different* sites covered by both system and copy. More precisely, suppose that at time t, the activity has covered σ_1 and σ_2 different sites in R_1 and R_2 respectively. Then, the function $\bar{n}_{cov}(t)$ is given by

$$\bar{n}_{cov}(t) = \langle \sigma_1 + \sigma_2 - \sigma_{1,2} \rangle \tag{17}$$

where $\sigma_{1,2}$ represents the number of sites covered in both systems (i.e. the *covering overlap* between system and copy). In the case of the ring, for large N and $t \ll N$, the overlap on the r.h.s. of equation (17) is empty ($\sigma_{1,2} \equiv 0$) and equation (16) reduces to equation (9). In the case of the BS model instead, this intersection cannot be empty even in the thermodynamic limit. As a consequence, the exponents predicted from equation (15) have to be considered as an upper bound for those observable in systems with non-trivial correlations.

3. Damage spreading in the BS Model

As mentioned above, in its simplest version the BS model describes an ecosystem as a collection of N species on a 1D lattice. To each species corresponds a fitness described by a number f

between 0 and 1. For simplicity, one considers periodic boundary conditions. The initial state of the system is defined by assigning to each site j a random fitness f_j chosen from a uniform distribution. The dynamics proceeds in three basic steps.

(1) Find the site with the absolute minimum fitness on the lattice (the active site) and its two nearest neighbours.
(2) Update the values of their fitnesses by assigning to them new random numbers from a uniform distribution.
(3) Return to step 1.

After an initial transient that will be of no interest to us here, a non-trivial critical state is reached. This critical state, characterised by its statistical properties, can be understood as the *fluctuating balance* between two competing 'forces'. Indeed, while the random assignation of the values, together with the coupling, acts as an entropic disorder, the choice of the minimum acts as an ordering force. As a result of this competition, at the stationary state the majority of the f_j have values above a certain threshold $f_c = 0.667\,02(1)$ [2]. In other words, the distribution function of the f_j's is given by

$$\eta_1(f) = \frac{1}{1-f_c}\Theta(f - f_c) \tag{18}$$

where $\Theta(f)$ is the Heaviside function. Only a few will be below f_c, namely those belonging to the running avalanche (see [2, 12] for a detailed discussion). Proceeding by analogy with the previous cases, once the system is at the critical state we produce two identical copies B_1 and B_2 and find the minimum (the active site). Then, in B_2 we swap the value of the minimum fitness with the fitness of some other site chosen at random (note that if N is big enough, the fitness in the other site will certainly be above threshold). After that, the evolution of the Hamming distance given by equation (6) is studied. In the evolution of both system and copy the same random numbers are used. Here, the length of the jumps in the position of the active site follows a power-law distribution given by equation (13) with $\beta \sim 3.23$ [2]. At variance with the case of the ring discussed above, we cannot expect the behaviour shown in equations (9) and (15). Indeed, in the BS model the jumps posses strong spatial correlations, with large probability of returning to sites already covered in previous timesteps. As a consequence, the behaviour of the number of different sites covered in one single system cannot be given by equation (15) but leads to $\sigma(t) \sim t^{\mu}$ for $t \gg 1$ [2], with $\mu = 0.4114 \pm 0.0020$ [13]. Moreover, as we consider the two copies B_1 and B_2 we realize that the two systems are also strongly correlated initially and consequently, one obtains an even smaller exponent† leading to

$$\langle D(t)\rangle \sim t^{\alpha=0.32}. \tag{19}$$

As we mentioned at the end of section 2, the behaviour of equation (19) can be understood in the framework of equation (16). In fact, the decrease in the value of α is given by the appearance of $\sigma_{1,2} \neq 0$ in equation (16). This is due to two phenomena. On the one hand we have avalanches. Indeed, when producing the initial perturbation in B_2, we will still have the old avalanche from which we have taken away the active site and put it somewhere else. This newly placed active site will start a new avalanche somewhere else but the activity will soon have to go back to the old (unfinished) avalanche. In the meantime, in B_1 the old avalanche has had some development. Therefore, chances to increase the number of identical sites in both systems are built in from the very beginning and the increase in $D(t)$ is slower.

† A new set of measurements, to be presented elsewhere [11], furnish an extrapolation for big N of the exponent α to be close to 0.4.

On the other hand, the fact that we are using the same sequence of random numbers implies that, if the system is big enough, the absolute minimum in one system is also the absolute minimum in the copy. Therefore, if the absolute minimum is among those sites which have not yet been covered by the activity, the three terms in the r.h.s. of equation (17) will have the same behaviour. If the minimum is instead one of the newer values put in the system after perturbation, its position on the lattice may be different in the two replica but the three functions in the r.h.s. of equation (17) grow slowly or even do not grow at all. This observation is confirmed by the irregular behaviour of $D(t)$ in just one single realization. In fact, it is the averaging procedure that finally produces a smooth curve. It should be noted that, as it is clear from its definition, the behaviour of the intersection is strongly correlated to the behaviour of the other two sets and therefore the average in the r.h.s. cannot be split into the sum of the independent averages. As a consequence we should expect a smaller exponent with respect to the one obtained for $\sigma(t)$.

The initial distance can be computed using equation (7). To do this we need the distribution function of the value of the minimum. Extensive numerical simulations indicate that this distribution can be approximated by

$$\eta_2(f) = (3 - \tfrac{9}{2}f)\Theta(\tfrac{2}{3} - f) \tag{20}$$

where the threshold has been put equal to $\frac{2}{3}$. Inserting (20) and (18) in equation (7) we obtain $\langle D(1)\rangle \sim \frac{11}{9N}$. Since $\langle D(t \to \infty)\rangle$ takes into account all sites on the same footing, this saturation value can be obtained from equation (10) with $\rho_1 = \rho_2 = \eta_1$. The distribution η_1 comes from equation (18), and the saturation value is $\langle D(t \to \infty)\rangle \sim \frac{1}{9}$. Therefore, as in the case of the ring, the saturation value does not depend on the size of the system while the initial distance does. Thus, the normalized distance reaches a plateau that must scale with N, as our numerical simulations show (see figure 2).

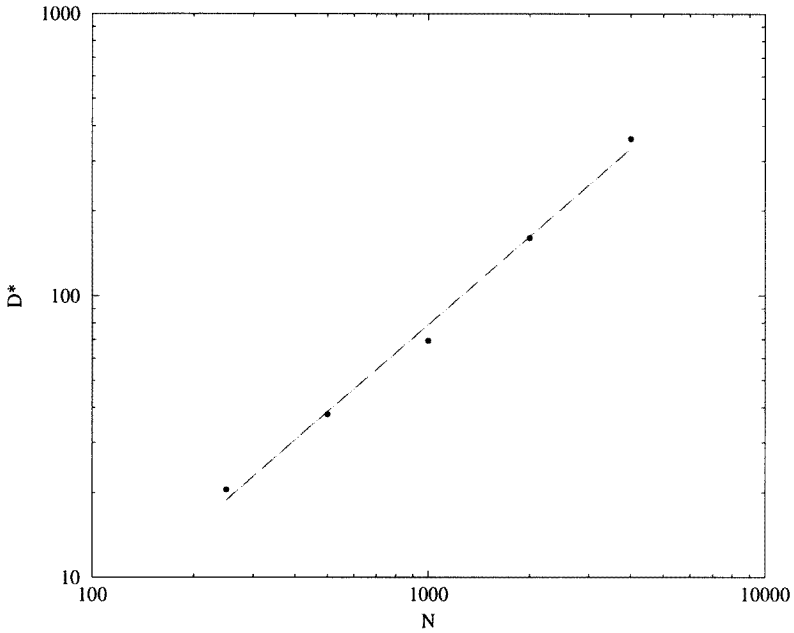

Figure 2. Scaling of the long-time plateau $D^* = \frac{\langle D(\infty)\rangle}{\langle D(1)\rangle}$ as a function of the number of sites N for the BS model. The best fit yields an exponent of 1.03(4).

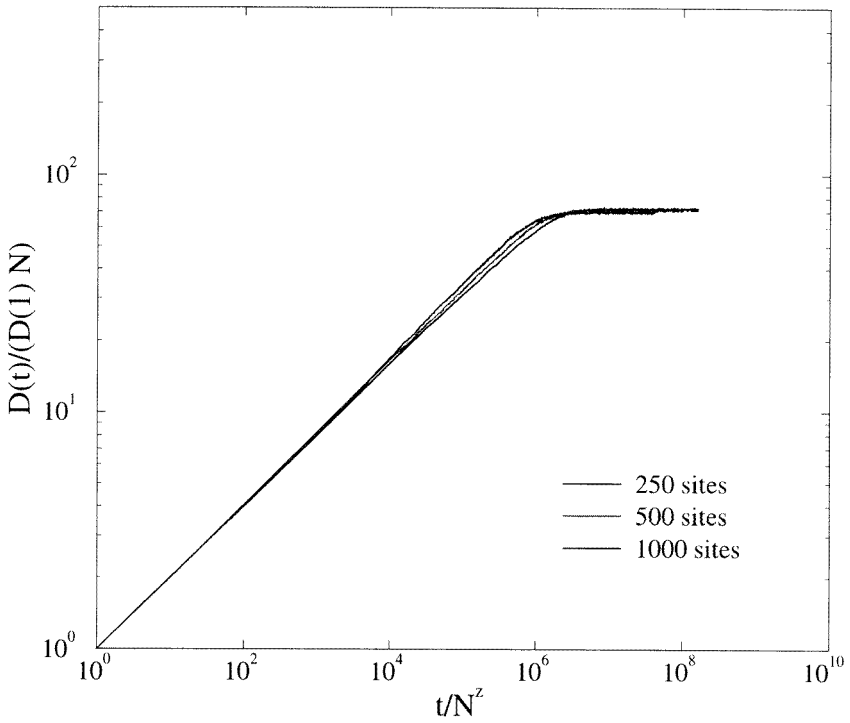

Figure 3. Collapse plot of the data for the evolution of the distance, in the BS case. The best fit yields $\alpha = 0.32(3)$ and $z = 3.0(2)$.

Coming back to the dynamical exponent z, we find that it still follows the prediction $z = 1/\alpha$ as in the case of the ring. For the BS model one obtains, following the above described prescription, $z \sim 3.12$, instead of $z \sim 1.6$ as determined in [1]. This value $z \sim 3.12$ coincides reasonably well with the one obtained from the collapse plot (figure 3). The reason for the discrepancy between our results and those presented in [1] can be traced back to the effects of time-rescaling on the (normalized) growth function. Indeed, let us assume we use a different timescale, and consider the case in which we make a measure of $D(t)$ every ν timesteps (instead of every timestep), where ν is distributed according to a certain function $P(\nu)$. The rescaled distance $\tilde{D}(t)$ will be given by

$$\tilde{D}(t) = \int \mathrm{d}\nu \, P(\nu) \, D((t-1)\langle\nu\rangle + \nu) \tag{21}$$

where $\langle\nu\rangle = \int \mathrm{d}\nu \nu P(\nu)$ is the average number of timesteps between two consecutive measurements. The choice made in [1] corresponds to $P(\nu) = \delta(\nu - N)$. It is easy to see that, in this case, the growth exponent for $\tilde{D}(t)$ is still α, but the measured dynamical exponent is given by $1/\alpha - 1$. In principle, one could imagine more complicated distributions $P(\nu)$ for the measuring time. In particular, if $P(\nu)$ did not have a finite first moment (as would be the case, for instance, if $P(\nu)$ corresponded to the avalanche distribution) equation (21) would yield $\tilde{D}(2) \approx N$, i.e. the rescaled distance would saturate almost immediately†.

At this point, it is worth discussing what happens in higher dimensions. The high-dimensional BS model has been extensively studied in [14], where the behaviour of the

† Since, for finite N every distribution has a cut-off, and therefore every moment is finite, this line of reasoning applies only in the limit $N \to \infty$. A more detailed analysis of this point will be presented elsewhere [11].

exponent μ for the growth of the quantity $\sigma(t)$ has been computed until the mean-field regime $\mu = 1$ was reached. In the framework of the damage spreading, taking into account the correlations as discussed above, one expects the exponent α to follow a similar pattern, reaching the value $\alpha = 1$ in the mean-field case. These mean-field results can also be obtained in the random-neighbours case [15]. However, from the point of view of damage spreading, the random nearest-neighbour case presents a complication. There is an ambiguity in the choice of the neighbours. Indeed, their absolute positions on the lattice should be either the same in the two copies of the system or taken at random in an uncorrelated fashion. In both cases, each one of the two copies will behave normally, but the behaviour of the distance will be completely different. Indeed, in the first case the distance between the two systems will never grow, while in the latter case the behaviour of the distance resembles that of the ring with uniformly distributed jumps.

4. Conclusions

Summarizing, we have shown that the power-law behaviour of the distance equation (2) originates in the behaviour of the mean-squared distance covered by the activity. This relationship has several consequences. First, one has $\alpha \leqslant 1$. Secondly, the internal correlations of the jumps, governed by equation (13), together with the strong correlations between the two copies, can severely slow down the growth of $\langle D(t) \rangle$. This leads, in turn, to exponents for the distance that are smaller than those predicted by equation (15). Since an analytic derivation of the exponents characterising the critical properties of the BS model is still lacking, our work had to be based, in part, on numerical results. From that starting point, after rewriting equation (2) in the more appropriate form given by equations (16), (17), we have been able to shed some light on the mechanisms leading to equation (2). In this framework, the reason for the appearance of a plateau can be easily understood and moreover, a prediction for its value can be made.

As a final point, we would like to emphasize that our analysis assumes that the distribution of the variables f does not change during the measurement of $\langle D(t) \rangle$ neither in the BS case nor in the case of the ring (we exclude the transient). This need not be so, that is one can ask oneself what happens in critical systems in which there is no steady distribution. This question is currently being addressed.

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